CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[12] Noncommutative gradings, language theory and free products.

1. Coproducts

All here is stated within the same category $\mathcal{C}$.

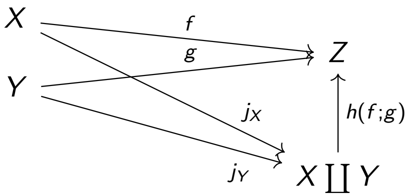


Figure: Coproduct $(j_X, j_Y; X \coprod Y)$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\ h(f;g) \in Hom(X \coprod Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{1}$$

Coproducts: Sets

All here is stated within the category **Set**.

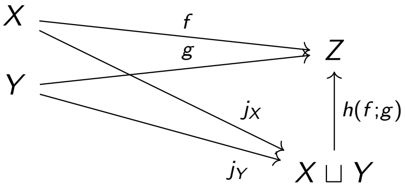


Figure: Coproduct $(j_X, j_Y; X \sqcup Y)$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists !\, h(f;g) \in Hom(X \sqcup Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{2}$$

Coproducts: Vector Spaces or modules

All here is stated within the same category $\mathbf{k}-\mathbf{Vect}$ (or $\mathbf{k}-\mathbf{Mod}$ if $\mathbf{k}$ is a ring).

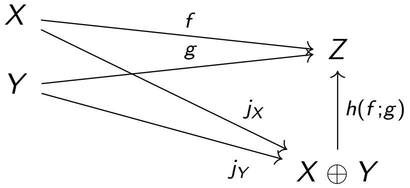


Figure: Coproduct $(j_X, j_Y; X \oplus Y)$ here $h(f;g) = f \oplus g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X \oplus Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{3}$$

Coproducts: $\mathbf{k} - \mathbf{CAAU}$

All here is stated within the same category $\mathbf{k} - \mathbf{CAAU}$.

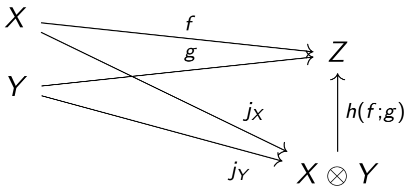


Figure: Coproduct $(j_X, j_Y; X \otimes Y)$ here $h(f;g) = f \otimes g$.

$$
\begin{aligned}
&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\
&\big(\exists!\, h(f;g) \in Hom(X \otimes Y, Z)\big)\\
&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)
\end{aligned}
\tag{4}
$$

Coproducts: Augmented $\mathbf{k} - \mathbf{AAU}$

All here is stated within the same category *Augmented* $\mathbf{k} - \mathbf{AAU}$.

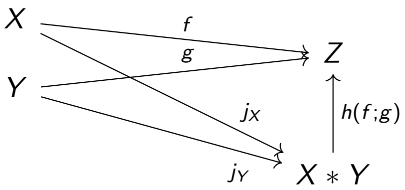


Figure: Coproduct $(j_X, j_Y; X * Y)$ here $h(f;g) = f * g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X * Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{5}$$

StackExchange Search on Mathematics…

MATHEMATICS

Home

Questions

Tags

Users

Unanswered

How to construct the coproduct of two (non-commutative) rings

Ask Question

Asked 7 years, 2 months ago Active 1 year, 9 months ago Viewed 3k times

20

17

How to construct/describe the coproduct of two - not necessarily commutative - rings R and S?

This in category **Ring** having as objects rings with a unit and as arrows unitary ringhomomorphisms.

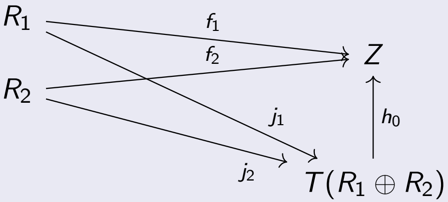
I thought of firstly constructing monoid M as coproduct of the underlying monoids $U(R)$ and $U(S)$ where $U : \mathbf{Ring} \to \mathbf{Mon}$ denotes the forgetful functor, and then secondly taking the ring $\mathbb{Z}[M]$ free over monoid M, but still have my doubts. If the rings have finite coprime characteristics then the coproduct should be the trivial ring, so something is wrong.

Can you give me a description of the coproduct (including its injections)? Thank you in advance.

ring-theory category-theory

The MSE question

1. We now turn to MSE question 625874
   "How to construct the-coproduct of two non commutative rings"
   https://en.wikipedia.org/wiki/Category_of_rings
   https://math.stackexchange.com/questions/625874
2. We address the question for $\mathcal{C} = \mathbf{Ring}$, the category of rings.
3. For R_1, R_2 two rings, we start with $T(R_1 \oplus R_2)$
4. So, with two morphisms $f_i : \; R_i \to Z, \; i = 1, 2$, we have



5. Rings are $\mathbb{Z}$-algebras. In order to get the tensors more readable, we color use colors (red for R_1, blue for R_2). The tensor algebra $T(R_1 \oplus R_2)$ reads

$$\begin{aligned}
&\mathbb{Z} \oplus R_1 \oplus R_2 \\
&\oplus (R_1 \otimes_{\mathbb{Z}} R_1) \oplus (R_1 \otimes_{\mathbb{Z}} R_2) \oplus (R_2 \otimes_{\mathbb{Z}} R_1) \oplus (R_2 \otimes_{\mathbb{Z}} R_2) \\
&\oplus (R_1 \otimes_{\mathbb{Z}} R_1 \otimes_{\mathbb{Z}} R_1) \oplus (R_1 \otimes_{\mathbb{Z}} R_1 \otimes_{\mathbb{Z}} R_2) \oplus (R_1 \otimes_{\mathbb{Z}} R_2 \otimes_{\mathbb{Z}} R_1) \\
&\oplus (R_1 \otimes_{\mathbb{Z}} R_2 \otimes_{\mathbb{Z}} R_2) \\
&\oplus (R_2 \otimes_{\mathbb{Z}} R_1 \otimes_{\mathbb{Z}} R_1) \oplus (R_2 \otimes_{\mathbb{Z}} R_1 \otimes_{\mathbb{Z}} R_2) \oplus (R_2 \otimes_{\mathbb{Z}} R_2 \otimes_{\mathbb{Z}} R_1) \\
&\oplus (R_2 \otimes_{\mathbb{Z}} R_2 \otimes_{\mathbb{Z}} R_2) \\
&\oplus \ldots
\end{aligned} \tag{6}$$

6. We see that tensors are of type

$$\{1, r, b, rr, rb, br, bb, rrr, rrb, rbr, rbb, brr, brb, bbr, bbb\}$$

7. Now comes to the rescue the new noncommutative grading (see paragraph "G-graded rings and algebras", in [24]).

The MSE question/3

8. In [24] algebras $\mathcal{A} = \oplus_{s\in G}\mathcal{A}_s$ when G is a group or a monoid (be it commutative of not) with the very natural condition $\mathcal{A}_s\mathcal{A}_t \subset \mathcal{A}_{st}$
9. A tensor of $T_n(R_1 \oplus R_2)$ (mind that this tensor can have 1_{R_i} as factors) of type $w \in \{r, b\}^*$ will be of the form

$$x_1 \otimes \ldots \otimes x_n \tag{7}$$

   where, for all $k \leq n$, $x_k \in R_{i(w[k])}$ where $i = 1$ if $w[k] = r$ and $i = 2$ if $w[k] = b$.
10. Likewise (and in general, because the at first the construction is linear in data) with $V = \oplus_{a\in A} V_a$, we have $T(V) = \oplus_{w\in A^*} T_w(V)$ and $T(V)$ is graded over A^*.
11. Returning to the original problem and notations of slide 10, we see that generically, we have $x \otimes y \equiv xy$ (computed in R_1) and $x \otimes y \equiv xy$ (computed in R_2).

The MSE question/4

12. We have then a very natural rewrite rule ($Rules1$)

$$P \otimes \underbrace{x_i \otimes x_{i+1}}_{same\ \ colour\ \ i \in \{r,b\}} \otimes S \to P \otimes \underbrace{x_i x_{i+1}}_{computed\ \ in\ \ R_i} \otimes S \tag{8}$$

13. So, only "count" the reduced "alternating tensors" (which cannot be reduced) i.e. of type $w \in (rb)^* \sqcup (br)^* \sqcup b(rb)^* \sqcup r(br)^*$
14. A second reduction rule occurs ($Rules2$) it is

$$P \otimes 1_{R_i} \otimes S \to P \otimes 1_{T(R_1 \oplus R_2)} \otimes S \tag{9}$$

15. Then, we obtain

$$R_1 * R_2 := T(R_1 \oplus R_2)/Rules \tag{10}$$

The MSE question/5

16. For example with $\mathbf{k}$ a (commutative) ring and $R_1 = \mathbf{k}[X]$, $R_2 = \mathbf{k}[Y]$, we get $R_1 * R_2 \simeq \mathbf{k}\langle X, Y\rangle$ whereas $R_1 \otimes R_2 \simeq \mathbf{k}[X, Y]$.
17. As remarked Darij Grinberg, in the discussion of the MSE question, it can happen that the factors may not embed in $R_1 * R_2$ (precisely through j_1, j_2, see slide 10).
18. In fact, this already happens for the category **CRing**, where the (free) coproduct is the tensor product i.e. where the product is given by

$$(u_1 \otimes v_1).(u_2 \otimes v_2) = u_1 v_1 \otimes u_2.v_2 \tag{11}$$

19. For example $\mathbb{Z}/3.\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2.\mathbb{Z} = \{0\}$. This is due to torsion as

$$u \otimes v = (3-2)(u \otimes v) = 3(u \otimes v) - 2(u \otimes v) = \\ 3u \otimes v - u \otimes 2v = 0$$

a similar computation shows that $\mathbb{Z}/3.\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}/2.\mathbb{Z} = \{0\}$ as $x = (3j_1(1) - 2j_2(1))x = 0$

StackExchange Search on Mathematics…

Home
Questions
Tags
Users
Unanswered

$U(R) \to U(\mathbb{Z}[U(R) \sqcup U(S)]) \leftarrow U(S)$ lift to ring homomorphisms $R \to \mathbb{Z}[U(R) \sqcup U(S)]/I \leftarrow S$.

Share Cite Edit Follow Flag edited Jan 3 '14 at 15:27 answered Jan 3 '14 at 14:29

Martin Brandenburg
134k ● 14 ■ 219 ▲ 403

Is what you call 'algebraic structure' the same thing as 'algebraic system (of a given type)' mentioned in CWM on page 120? By the way, you are of great help to me on this site. Not only in this question. Tibi gratias ago! – drhab Jan 3 '14 at 14:44

Yes, exactly. CWM keeps this quite short (although very concise), you can find more about algebraic structures in texts about *universal algebra*. – Martin Brandenburg Jan 3 '14 at 14:46

1 I don't think R and S *embed* into your coproduct in the sense of injective maps. Try $S = 0$ and $R \neq 0$. – darij grinberg Jan 3 '14 at 15:25

@Darij: You are right, thank you. – Martin Brandenburg Jan 3 '14 at 15:27

1 It doesn't make sense to talk about $R \cap S$. And yes, R and S are completely isolated. – Martin Brandenburg Jun 25 '15 at 12:59

Add a comment | Show **7** more comments

Your Answer

The MSE question/6

20. We remark that if $R_i = \mathbb{Z}.1_{R_i} \oplus R_i^+$ and R_i^+ is closed by products (this amount to saying that $R \to \mathbb{Z}.1_{R_i}$ is a ring morphism i.e. that R_i are augmented).
21. In this case (R_i are augmented), the making of $R_1 * R_2$ is
    1. Compute $T(R_1^+ \sqcup R_2^+)/Rules1 =: (R_1 * R_2)_+$
    2. Adjoint a unit and obtain

    $$(R_1 * R_2) := (R_1 * R_2)_+ \oplus \mathbb{Z}$$

    as an augmented ring.

Concluding remarks and perspectives

1. We constructed free products of rings.
2. This construction holds mutatis mutandis for algebras (and then rings as a particular case considering that $\textbf{Ring} = \mathbb{Z} - \textbf{AAU}$)
3. These constructions are useful for twisted actions as differential polynomials and Ore algebras. They will be the object of a forthcoming CCRT.
4. In this CCRT, we will explore also more general twisted actions as coloured tensor products in appropriate categories.

Thank you for your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

[1] N. BOURBAKI, *Algebra I (Chapters 1-3)*, Springer 1989.

[2] N. Bourbaki, *Algèbre, Chapitre 8*, Springer, 2012.

[3] N. Bourbaki.– *Lie Groups and Lie Algebras, ch 1-3*, Addison-Wesley, ISBN 0-201-00643-X

[4] P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

[5] M. Deneufchâtel, GD, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[6] Jean Dieudonné, *Foundations of Modern Analysis*, Volume 2, Academic Press; 2nd rev edition (January 1, 1969)

[7] G. Duchamp, D.Krob, *Free partially commutative structures*, Journal of Algebra, 156 , 318-359 (1993)

[8] GD, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[9] GD, Darij Grinberg, Vincel Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, ArXiv:2009.10970 [math.QA] (Wed, 23 Sep 2020)

[10] Gérard H. E. Duchamp, Christophe Tollu, Karol A. Penson and Gleb A. Koshevoy, *Deformations of Algebras: Twisting and Perturbations* , Séminaire Lotharingien de Combinatoire, B62e (2010)

[11] GD, Nguyen Hoang-Nghia, Thomas Krajewski, Adrian Tanasa, *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*, Advances in Applied Mathematics 51 (2013) 345–358.

[12] K.T. Chen, R.H. Fox, R.C. Lyndon, *Free differential calculus, IV. The quotient groups of the lower central series*, Ann. of Math. , 68 (1958) pp. 81–95

[13] V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal(*$\bar{\mathbb{Q}}/\mathbb{Q}$*)*, Leningrad Math. J., 4, 829-860, 1991.

[14] M.E. Hoffman, *Quasi-shuffle algebras and applications*, arXiv preprint arXiv:1805.12464, 2018

[15] H.J. Susmann, *A product expansion for Chen Series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrns and Lindquist (eds). 323-335, 1986

[16] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[17] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997).

[18] Szymon Charzynski and Marek Kus, *Wei-Norman equations for a unitary evolution*, Classical Analysis and ODEs, J. Phys. A: Math. Theor. 46 265208

[19] Rimhac Ree, *Lie Elements and an Algebra Associated With Shuffles*, Annals of Mathematics Second Series, Vol. 68, No. 2 (Sep., 1958)

[20] G. Dattoli, P. Di Lazzaro, and A. Torre, *$SU(1,1), SU(2)$, and $SU(3)$ coherence-preserving Hamiltonians and time-ordering techniques. Phys.* Rev. A, 35:1582–1589, 1987.

[21] J. Voight, *Quaternion algebras*, `https://math.dartmouth.edu/~jvoight/quat-book.pdf`

[22] Adjuncts in nlab. `https://ncatlab.org/nlab/show/adjunct`

[23] M. van der Put, M. F. Singer.– *Galois Theory of Linear Differential Equations*, Springer (2003)

[24] Graded rings, see “Graded Rings and Algebras”, see “G-graded rings and algebras” in
https://en.wikipedia.org/wiki/Graded_ring

[25] How to construct the coproduct of two non-commutative rings
https://math.stackexchange.com/questions/625874

[26] Definition of (commutative) free augmented algebras
https://mathoverflow.net/questions/352726

[27] Closed subgroup (Cartan) theorem without transversality nor Lipschitz condition within Banach algebras
https://mathoverflow.net/questions/356531

[28] Definition of augmented algebras (general)
https://ncatlab.org/nlab/show/augmented+algebra

[29] Coproduct of two non commutative rings
https://math.stackexchange.com/questions/625874